

ROTATIONAL SURFACES IN ISOTROPIC SPACES SATISFYING WEINGARTEN CONDITIONS

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Abstract. In this paper, we study the rotational surfaces in the isotropic 3-space \mathbb{I}^3 satisfying Weingarten conditions in terms of the relative curvature K (analogue of the Gaussian curvature) and the isotropic mean curvature H . In particular, we classify such surfaces of linear Weingarten type in \mathbb{I}^3 .

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1 Introduction

The work of surfaces with special properties in the isotropic 3-space \mathbb{I}^3 has important applications in several applied sciences, e.g., computer science, Image Processing, architectural design and microeconomics, see [3, 4, 6, 8], [29]-[31].

Differential geometry of isotropic spaces have been introduced by K. Strubecker [37], H. Sachs [32]-[34], D. Palman [27] and others.

I. Kamenarovic ([17, 18]), B. Pavkovic ([28]), Z. M. Sipus ([35]) and M.E. Aydin ([1, 2]) have studied some classes of surfaces in \mathbb{I}^3 .

On the other hand, let \mathcal{M} be a regular surface of a Euclidean 3-space \mathbb{R}^3 . For general references on the geometry of surfaces see [12, 15].

Denote ∇ the Levi-Civita connection of \mathbb{R}^3 and N the normal vector field to \mathcal{M} . Then the operator given by

$$S(v) = -\nabla_v N,$$

is called the *shape operator*, where v is a tangent vector field to \mathcal{M} . It measures how \mathcal{M} bends in different directions. The eigenvalues of S are called the *principal curvatures* denoted by κ_1 and κ_2 .

The arithmetic mean of the principal curvatures are called the *mean curvature*, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$. The *Gaussian curvature* is defined by $K = \kappa_1 \kappa_2$.

A surface \mathcal{M} in \mathbb{R}^3 is called a *Weingarten surface* (*W-surface*) if it satisfies the following non-trivial functional relation

$$\phi(\kappa_1, \kappa_2) = 0$$

for a smooth function ϕ of two variables. The above relation implies the following

$$\delta(K, H) = 0,$$

which is the equivalent to the vanishing of the corresponding Jacobian determinant, i.e. $\left| \frac{\partial(K,H)}{\partial(u,v)} \right| = 0$ for a coordinate pair (u, v) on \mathcal{M} .

If \mathcal{M} fulfills the following condition

$$c_1 H + c_2 K = c_3, \quad c_i \in \mathbb{R}, \quad (c_1, c_2, c_3) \neq (0, 0, 0), \quad i = 1, 2, 3,$$

then it is called a *linear Weingarten surface (LW-surface)*. In the particular case $c_1 = 0$ (resp. $c_2 = 0$), the LW-surfaces are indeed the surfaces with constant Gaussian curvature (resp. mean curvature). These phenomenal surfaces have been studied by many geometers in various ambient spaces, see [14, 20], [22]–[24], [26], [38].

The motivation of the present paper is to study Weingarten surfaces, in particular Weingarten rotational surfaces, in the isotropic 3-space \mathbb{I}^3 which is one of the Cayley–Klein spaces.

Most recently, M.E. Aydin ([2]) classified the helicoidal surfaces in \mathbb{I}^3 , which are natural generalization of the rotational surfaces, with constant curvature and analyzed some special curves on such surfaces.

In the present paper, we provide that the rotational surfaces in \mathbb{I}^3 are evidently Weingarten ones. Then we classified LW-rotational surfaces in \mathbb{I}^3 satisfying the following relation

$$K = m_0 H + n_0, \quad m_0, n_0 \in \mathbb{R},$$

in which K is the relative curvature and H isotropic mean curvature.

2 Preliminaries

The isotropic 3-space \mathbb{I}^3 is obtained from the 3-dimensional projective space $P(\mathbb{R}^3)$ with the absolute figure which is an ordered triple (p, l_1, l_2) , where p is a plane in $P(\mathbb{R}^3)$ and l_1, l_2 are two complex-conjugate straight lines in p (see [35]). The homogeneous coordinates in $P(\mathbb{R}^3)$ are introduced in such a way that the *absolute plane* p is given by $x_0 = 0$ and the *absolute lines* l_1, l_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. The intersection point $P(0 : 0 : 0 : 1)$ of these two lines is called the *absolute point*. The group of motions of \mathbb{I}^3 is a six-parameter group given in the normal form (in affine coordinates) $\mathbf{x} = \frac{x_1}{x_0}$, $\mathbf{y} = \frac{x_2}{x_0}$, $\mathbf{z} = \frac{x_3}{x_0}$ by

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}', \mathbf{y}', \mathbf{z}') : \begin{cases} \mathbf{x}' = c_1 + \mathbf{x} \cos c_2 - \mathbf{y} \sin c_2, \\ \mathbf{y}' = c_3 + \mathbf{x} \sin c_2 + \mathbf{y} \cos c_2, \\ \mathbf{z}' = c_4 + c_5 \mathbf{x} + c_6 \mathbf{y} + \mathbf{z}, \end{cases} \quad (2.1)$$

for $c_1, \dots, c_6 \in \mathbb{R}$.

Such affine transformations are called *isotropic congruence transformations* or *i-motions*.

Consider the points $p_1 = (x_1, x_2, x_3)$ and $p_2 = (y_1, y_2, y_3)$. The *isotropic distance*, so-called *i-distance* of two points p_1 and p_2 is defined by

$$\|p_1 - p_2\|_i = \left((y_1 - x_1)^2 + (y_2 - x_2)^2 \right)^{\frac{1}{2}}.$$

The i-metric is degenerate along the lines in \mathbf{z} -direction, and such lines are called *isotropic* lines.

Planes, circles and spheres. There are two types of planes in \mathbb{I}^3 ([29]-[31]).

(1) *Non-isotropic planes* are planes non-parallel to the \mathbf{z} -direction. In these planes we basically have an Euclidean metric: This is not the one we are used to, since we have to make the usual Euclidean measurements in the top view. An *i-circle* (of *elliptic type*) in a non-isotropic plane p is an ellipse, whose top view is an Euclidean circle. Such an i-circle with center $c_0 \in p$ and radius r is the set of all points $x \in p$ with $\|x - c_0\|_i = r$.

(2) *Isotropic planes* are planes parallel to the \mathbf{z} -axis. There, \mathbb{I}^3 induces an isotropic metric. An *i-circle* (of *parabolic type*) is a parabola with \mathbf{z} -parallel axis and thus it lies in an isotropic plane

An i-circle of parabolic type is not the iso-distance set of a fixed point, but it may be seen as a curve with constant isotropic curvature: A curve α in an isotropic plane P (without loss of generality we set $P : \mathbf{y} = 0$) which does not possess isotropic tangents can be written as graph $\mathbf{z} = f(\mathbf{x})$. Then, the *i-curvature* of α at $\mathbf{x} = s_0$ is given by the second derivative $\kappa_i(s_0) = f''(s_0)$. For an i-circle of parabolic type f is quadratic and thus κ_i is constant.

There are also two types of *isotropic spheres*. An *i-sphere of the cylindrical type* is the set of all points $x \in \mathbb{I}^3$ with $\|x - c_0\|_i = r$. Speaking in an Euclidean way, such a sphere is a right circular cylinder with \mathbf{z} -parallel rulings; its top view is the Euclidean circle with center c_0 and radius r . The more interesting and important type of spheres are the *i-spheres of parabolic type*,

$$\mathbf{z} = \frac{A}{2} (\mathbf{x}^2 + \mathbf{y}^2) + B\mathbf{x} + C\mathbf{y} + D, \quad A \neq 0.$$

From an Euclidean perspective, they are paraboloids of revolution with \mathbf{z} -parallel axis. The intersections of these i-spheres with planes p are i-circles. If p is non-isotropic, then the intersection is an i-circle of elliptic type. If p is isotropic, the intersection curve is an i-circle of parabolic type.

Curvature theory of surfaces. A surface \mathcal{M} immersed in \mathbb{I}^3 is called *admissible* if it has no isotropic tangent planes. We restrict our framework to admissible regular surfaces. For such a surface \mathcal{M} , the coefficients E, F, G of its first fundamental form are calculated with respect to the induced metric.

The normal field of \mathcal{M} is always the isotropic vector. The coefficients L, M, N of the second fundamental form of \mathcal{M} are calculated with respect to the normal field of \mathcal{M} (for details, see [33], p. 155).

The *relative curvature* (so called *isotropic Gaussian curvature*) and *isotropic mean curvature* are defined by

$$K = \frac{LM - N^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2EG - F^2}. \quad (2.2)$$

3 LW-rotational surfaces in \mathbb{I}^3

Let us consider the i-motions given by (2.1), then the Euclidean rotations in the isotropic space \mathbb{I}^3 is given by in affine coordinates

$$\begin{cases} \mathbf{x}' = c_1 + \mathbf{x} \cos c_2 - \mathbf{y} \sin c_2, \\ \mathbf{y}' = c_3 + \mathbf{x} \sin c_2 + \mathbf{y} \cos c_2, \\ \mathbf{z}' = \mathbf{z}, \end{cases}$$

where $c_i \in \mathbb{R}$.

Definition 3.1. Let α be a curve lying in the isotropic xz -plane given by $c(u) = (u, 0, g(u))$ where $g \in C^2$, $\frac{dg}{du} \neq 0$. By rotating the curve c around z -axis, we obtain that the rotational surface in \mathbb{I}^3 is of the form

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, g(u)). \quad (3.1)$$

Similarly when the profile curve α lies in the isotropic yz -plane, then the rotational surface in \mathbb{I}^3 is given by

$$\mathbf{X}(u, v) = (-u \sin v, u \cos v, g(u)). \quad (3.2)$$

Remark 3.1. The rotational surfaces given by (3.1) and (3.2) are locally isometric and thus we only consider the ones of the form (3.1).

Let \mathcal{M} be the rotational surface given by (3.1) in \mathbb{I}^3 . Then the nonzero components of first fundamental form of \mathcal{M} are calculated by induced metric from \mathbb{I}^3 as follows

$$E = 1, G = u^2. \quad (3.3)$$

The nonzero components of second fundamental form of \mathcal{M} are

$$L = g'', N = ug', \quad (3.4)$$

where $g' = \frac{dg}{du}$ and $g'' = \frac{d^2g}{du^2}$. From (2.2), (3.3) and (3.4), we get

$$K = \frac{1}{u}g'g'', H = \frac{1}{u}g' + g'', \quad (3.5)$$

which yields that the curvatures K and H depend only on the variable u , namely $\left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0$. In the sequel, we have the following result.

Theorem 3.1. Rotational surfaces in \mathbb{I}^3 are Weingarten surfaces.

We are also able to investigate the LW-rotational surfaces in \mathbb{I}^3 with the relation

$$K = m_0 H + n_0, \quad m_0, n_0 \in \mathbb{R}. \quad (3.6)$$

If $m_0 = 0$ in (3.6), then those reduce to ones with constant relative curvature. Thus we aim to obtain the LW-rotational surfaces in \mathbb{I}^3 with $m_0 \neq 0$.

The following result classifies the LW-rotational surfaces satisfying (3.6).

Theorem 3.2. *Let \mathcal{M} be a LW-rotational surface in \mathbb{I}^3 . Then one of the following holds*

(i) \mathcal{M} is of the form

$$\begin{cases} \mathbf{X}(u, v) = (u \cos v, u \sin v, g(u)), \\ g(u) = \frac{m_0}{2}u^2 \pm \frac{u}{2}\sqrt{c_1 + m_0^2 u^2} \pm c_2 \ln \left| 2m_0 \left(m_0 u + \sqrt{c_1 + m_0^2 u^2} \right) \right|, \\ c_1, c_2 \in \mathbb{R} \setminus \{0\}; \end{cases}$$

(ii) \mathcal{M} is an elliptic paraboloid from the Euclidean perspective, i.e.

$$\begin{cases} \mathbf{X}(u, v) = (u \cos v, u \sin v, g(u)), \\ g(u) = \frac{m_0}{2}u^2 + c_3, \quad c_3 \in \mathbb{R}; \end{cases}$$

(iii) \mathcal{M} is given by

$$\begin{cases} \mathbf{X}(u, v) = (u \cos v, u \sin v, g(u)), \\ g(u) = \frac{m_0}{2}u^2 \pm \frac{u}{2}\sqrt{c_1 + (m_0^2 + n_0)u^2} \pm \\ \pm \frac{c_1}{m_0^2 + n_0} \ln \left| 2 \left((m_0^2 + n_0)u + \sqrt{m_0^2 + n_0} \sqrt{c_1 + (m_0^2 + n_0)u^2} \right) \right|, \\ c_1 \in \mathbb{R}, \quad c_1 < 0. \end{cases}$$

Proof. Assume \mathcal{M} is a LW-rotational surface in \mathbb{I}^3 having the relation (3.6). Then, from (3.5), it follows

$$\frac{1}{u}g'g'' = m_0 \frac{g' + g''}{u} + n_0. \quad (3.7)$$

We have two cases:

Case a. $n_0 = 0$. Hence we can rewrite (3.7) as

$$g''(g' - m_0 u) - m_0 g' = 0. \quad (3.8)$$

If $g' = m_0 u$ in (3.8), then g' and m_0 vanish which is not possible. Then we have

$$g'' - \frac{m_0 g'}{g' - m_0 u} = 0. \quad (3.9)$$

By solving (3.9), we obtain

$$g(u) = \frac{m_0}{2}u^2 \pm \frac{u}{2}\sqrt{e^{2c_1} + m_0^2 u^2} \pm \frac{e^{2c_1}}{2m_0} \ln \left| 2m_0 \left(m_0 u + \sqrt{e^{2c_1} + m_0^2 u^2} \right) \right|,$$

$c_1 \in \mathbb{R}$, which gives the statement (i) of the theorem.

Case b. $n_0 \neq 0$. Then we have from (3.7)

$$g''(g' - m_0 u) - m_0 g' = n_0 u. \quad (3.10)$$

When $g' = m_0 u$, then $g(u) = \frac{m_0}{2} u^2 + c_2$, $c_2 \in \mathbb{R}$ and $n_0 = -m_0^2$. This implies the statement (ii) of the theorem.

Otherwise, we conclude from (3.10) that

$$g'' - \frac{m_0 g'}{g' - m_0 u} = \frac{n_0 u}{g' - m_0 u}. \quad (3.11)$$

After solving (3.11), we derive

$$g(u) = \frac{m_0}{2} u^2 \pm \frac{u}{2} \sqrt{-e^{2c_3} + (m_0^2 + n_0) u^2} \mp \frac{e^{2c_3}}{m_0^2 + n_0} \ln \left| 2 \left((m_0^2 + n_0) u + \sqrt{m_0^2 + n_0} \sqrt{-e^{2c_3} + (m_0^2 + n_0) u^2} \right) \right|,$$

$c_3 \in \mathbb{R}$. Therefore the proof is completed.

Example 3.1. Consider the elliptic paraboloid in \mathbb{I}^3 from the Euclidean perspective given by

$$\mathbf{X}(u, v) = (u \cos v, u \sin v, 0.25u^2), \quad (u, v) \in [0, 2\pi].$$

Then $K = 0.25$, $H = 1$, $m_0 = 0.5$ and $n_0 = -0.25$. We plot it as in Fig. 1.

Fig 1. *LW-rotational surface with $m_0 = 0.5$, $n_0 = -0.25$*

4 Rotational surfaces in \mathbb{I}^3 with $H/K = \text{const.}$

The authors in [7] introduced a new kind of curvature for the hypersurfaces of Euclidean n -spaces, called by amalgamatic curvature and explored its geometric meaning by proving an inequality related to the absolute mean curvature of the hypersurface. In the particular case $n = 3$, the amalgamatic curvature is indeed the harmonic ratio of the principal curvatures of any given surface, i.e., the ratio of the Gaussian curvature and the mean curvature.

By considering this argument, we can consider the rotational surfaces in \mathbb{I}^3 satisfying $H/K = \text{const.}$ Thus the statement (i) of Theorem 3.2 is indeed a classification of the rotational surfaces in \mathbb{I}^3 satisfying $H/K = \text{const.}$

Therefore, we have the following trivial result.

Corollary 4.1. *Let \mathcal{M} be a LW-rotational surface in \mathbb{I}^3 satisfying $H/K = \frac{1}{m_0}$, $m_0 \in \mathbb{R} \setminus \{0\}$. Then it is of the form*

$$\begin{cases} \mathbf{X}(u, v) = (u \cos v, u \sin v, g(u)), \\ g(u) = \frac{m_0}{2}u^2 \pm \frac{u}{2}\sqrt{c_1 + m_0^2 u^2} \pm c_2 \ln \left| 2m_0 \left(m_0 u + \sqrt{c_1 + m_0^2 u^2} \right) \right|, \\ c_1, c_2 \in \mathbb{R} \setminus \{0\}; \end{cases} \quad (4.1)$$

Example 4.1. *Take $\lambda_0 = 0.5$ and $c_1 = \ln 2$ in (4.1). Then we obtain a rotational surface in \mathbb{I}^3 with $H/K = 1$ given by*

$$\mathbf{X}(u, v) = \left(u \cos v, u \sin v, u^2 + u\sqrt{1 + u^2} + \ln \left| 2 \left(u + \sqrt{1 + u^2} \right) \right| \right),$$

where $u \in [0, 2\pi]$, $v \in [0, \pi/2]$. Then it can be plotted as in Fig. 2.

Fig 2. Rotational surface with $H/K = 1$

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